Hybrid-Hodge Matrix

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By applying the theory of differential forms to solve problems of wave propagation, we must solve sparse linear system defined by the insertion of constitutive laws through Hodge matrix. In this article we use two techniques of construction of the Hodge matrix, namely, the Galerkin method and Yee method. These techniques are used to construct a hybrid matrix having the best of both techniques for the treatment of medium changes. Numerical results will be presented to showed the performance of approaches to solve static problems in two-dimensional spaces.

***Index Terms*—Differential forms, Medium changes, Hodge matrix.**

I. Introduction

THE resolution of electromagnetic problems through the use of Differential Forms [1], introduces a sparse linear system defined by the insertion of the constitutive laws by Hodge matrices [2]. These systems include Hodge matrices, whose function is to map the degrees of freedom that represent the electric field to the electric flux density or magnetic field to the magnetic flux density.

There are two ways to construct Hodge matrices, one by geometric shape, which takes into account the size of the simplex involved in the mapping, this type of matrix is known as Yee-Hodge matrices, whose main characteristic is the fact that they are diagonal. But this type of matrix does not have a good performance due to its characteristics when we have extreme changes in media.

Another way to construct Hodge matrices is by using Galerkin approximation with the use Whitney functions as basis function and interpolation. For each p-simplex of the mesh, there is a Whitney p-form associated with it. The Galerkin-Hodge matrices has it's entries generated by integrals of Whitney p-forms on the p-simplex. These matrices aren't diagonal, defined positive, symmetric and sparse in simplicial meshes [2].

These matrices treat fine changes media, but by being sparse and non-diagonal typically have a full inverse, thus causing a high cost computational its inversion unlike a diagonal matrix [3].

This paper presents the formulation for the construction of a matrix which combines the best of the Yee-Hodge matrix and Galerkin-Hodge matrix. The idea is to obtain a matrix more sparse then the Galerkin-Hodge matrix, which will deal in an effective manner the changes media, in regions with discontinuity.

II. Hodge Matrix

We present now two ways to obtain Hodge matrices. The Yee-Hodge matrix will be obtained from purely geometrical way, through the topological information of the primal and dual meshes. The other will be achieved through the use of the Whitney p-forms and the Galerkin approximation, this is known as the Galerkin-Hodge matrix. And ending this section show a way to use the forms of construction of Hodge matrix to obtain a matrix having good performance for changes in the middle and with density under Galerkin-Hodge matrix.

A. Yee-Hodge Matrix

In the discretization of the domain is considered a mesh with 2d simplicial N_t triangles, N_e edges and N_n nodes. The Yee-Hodge matrices are constructed according to the ratio between the modules of the simplex involved in the primal and dual mesh.

The terms of the entries of the diagonal in the Yee-Hodge matrices will be calculated as follows

$$[\mathbf{M}(\alpha)]_{i,i} = \alpha|s_i^*|/|s_i| \quad (1)$$

where α denote a scalar fields ϵ (permittivity), ν (reluctivity), σ (conductivity), and s_i, s_i^* are the i-th primal and i-th dual simplex, respectively. And off diagonal, the entries are zero. Using the Kronecker delta, construction of the matrix can be written in general form

$$[\mathbf{M}(\alpha)]_{ij} = \alpha|s_i^*|/|s_j|\delta_{ij} \quad (2)$$

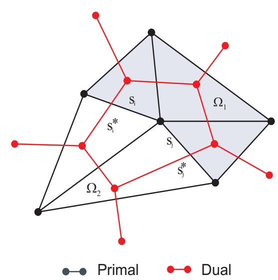

Fig. 1. Primal and dual mesh in two-dimensional domains.

Figure 1 shows the primal and dual meshes in a two-dimensional domain and primal 1-simplex s_i and dual 1-simplex s_i^*.

B. Galerkin-Hodge Matrix

The construction of the Hodge matrix by Galerkin method makes use of Whitney p-forms [4]. Considering once again a simplicial mesh in 2D, we can associate each p-simplex to a Whitney p-form.

The Hodge-Galerkin matrix is then defined as follows

$$[\mathbf{M}_p(\alpha)]_{ij} = \int_\Omega \alpha\omega_p^i.\omega_p^j d\Omega \tag{3}$$

where ω_p^i denotes the i-th Whitney p-form associated with the i-th p-simplex of the mesh.

C. Hybrid-Hodge Matrix

The principal idea is used the best of formulations, Yee-Hodge and Galerkin-Hodge for contruction a Hybrid-Hodge matrix.

In the region near the boundary of two media with different electromagnetic characteristics we use the Hodge-Galerkin formulation, due to its good performance for treatment in changes of medium. In the other regions used the formulation Yee-Hodge.

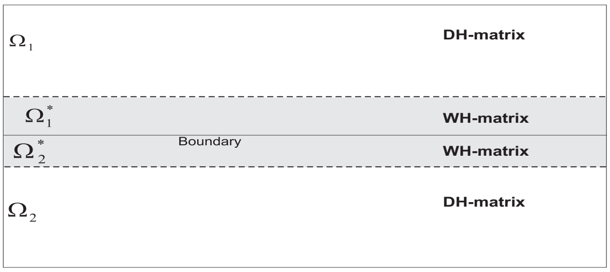

Fig. 2. Domain with two mediums and regions for constructions the Hybrid-Hodge matrix.

Figure 2 shows the domain with two mediums and the regions for constructions the Hybrid-Hodge matrix. The Hybrid-Hodge matrix is an matrix with sparsity between the sparsity of the matrices Yee-Hodge and Galerkin-Hodge with the precision of change media the Galerkin-Hodge matrix.

Equation (4) shows the formula for the construction of the Hybrid-Hodge matrix:

$$[\mathbf{M}_p(\alpha)]_{ij} = \begin{cases} \int_\Omega \alpha\omega_p^i.\omega_p^j d\Omega \text{ if } \Omega = \Omega_1^*, \Omega_2^* \\ \\ \alpha|s_i|/|s_j^*|\delta_{ij} \text{ other cases} \end{cases} \tag{4}$$

III. Numerical Results

The techniques presented in Session II, will now be used to solve a magnetostatic problem and their results will be compared. The problem of the electromagnet, with change of material it's a problem with a significant degree of complexity because it involves changing materials, and this choice is intended to test the efficiency of the Hodge operator in the Galerkin and Yee formulation.

Figure 3 shows the problem domain with particular circular region which uses the Hodge-Galerkin formulation for the construction of the Hodge matrix and the rest of the domain which uses the formulation Yee-Hodge.

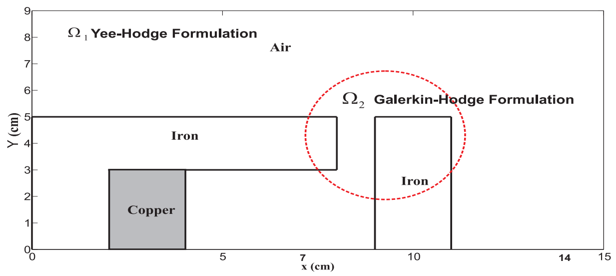

Fig. 3. Domain with two mediums and regions for constructions the Hybrid-Hodge matrix.

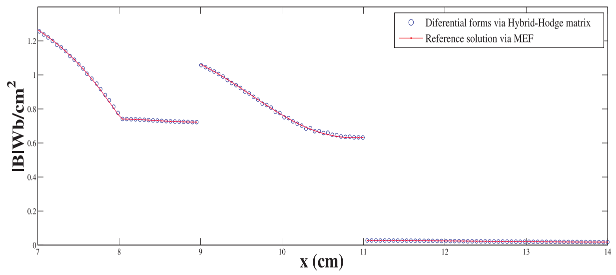

Fig. 4. Magnetic flux density evaluated by Differential Forms on the segment $y = 4$ and $7 \leq x \leq 14$, using Hybrid-Hodge matrix.

Figure 4 shows the distribution of the vector potential magnetite calculated from the Differential Forms using the Hybrid-Hodge matrix in all over computational domain, we observe that for this distribution consistency with the physical behaviour expected for the problem of electromagnet. The density of the hybrid matrix Hodge used in this problem was 0.00021%, about three times more sparse the the density of Glerkin-Hodge matrix equal to 0.00077%.

IV. Conclusion

In this paper we present the construction of the Hybrid-Hodge matrix and simulations showed a good performance of this matrix in the treatment of change of means. Full article at present convergence tests and details of the formulation.

Acknowledgment

This work was supported in part by the State of Minas Gerais Research Foundation - FAPEMIG, Brazil, State of Espirito Santo Research Foundation - FAPES, the Brazilian government agency - CAPES and National Council for Scientific and Technological Development - CNPq, Brazil.

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